Math 334 Test 3 KEY
Spring 2010
Section: 001

Sections 5.2-7.5 of Boyce and DiPrima

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Honor Code: I will not look at other students' papers as I am taking this exam.

Signature:

1. Find the first 5 terms in the series representation (about $x=0$)of the general solution of the following linear, second order ODE:

$$\left(2x^3-1\right)y''-6x^2y'+6xy=0. \tag{0.1}$$

15 points

Solution

Writing

$$y=y(x)=\sum_{n=0}^{\infty}a_nx^n=:\sum a_nx^n \tag{0.2}$$

we get (0.1) is

$$\begin{aligned}0&=\left(2x^3-1\right)y''-6x^2y'+6xy=\sum\left(2x^3-1\right)n\left(n-1\right)a_nx^{n-2}-6na_nx^2x^{n-1}+6xa_nx^n\\&=\sum 2n\left(n-1\right)a_nx^{n+1}-n\left(n-1\right)a_nx^{n-2}-6na_nx^{n+1}+6a_nx^{n+1}\\&=\sum 2n\left(n-1\right)a_nx^{n+1}-\left(n+3\right)\left(n+2\right)a_{n+3}x^{n+1}-6na_nx^{n+1}+6a_nx^{n+1}\\&=\sum\left(2\left(n^2-4n+3\right)a_n-\left(n+3\right)\left(n+2\right)a_{n+3}\right)x^{n+1}\\&=\sum\left(2\left(n-1\right)\left(n-3\right)a_n-\left(n+3\right)\left(n+2\right)a_{n+3}\right)x^{n+1}\end{aligned} \tag{0.3}$$

$$\Leftrightarrow$$

$$a_{n+3}=\frac{2\left(n-1\right)\left(n-3\right)}{\left(n+3\right)\left(n+2\right)}a_n,\ n=-1,0,1,2,\ldots$$

$$\Leftrightarrow$$

$$a_{n+2}=\frac{2\left(n-2\right)\left(n-4\right)}{\left(n+2\right)\left(n+1\right)}a_{n-1},\ n=0,1,2,\ldots.$$

So

$$
\begin{aligned}
a_2 &= a_{0+2} = \frac{2(0-2)(0-4)}{(0+2)(0+1)} a_{0-1} = 8a_{-1} = 8\cdot 0 = 0, \\
a_3 &= a_{1+2} = \frac{2(1-2)(1-4)}{(1+2)(1+1)} a_{1-1} = a_0, \\
a_4 &= a_{2+2} = \frac{2(2-2)(2-4)}{(2+2)(2+1)} a_{2-1} = 0\cdot a_1 = 0, \\
a_5 &= a_{3+2} = \frac{2(3-2)(3-4)}{(3+2)(3+1)} a_{3-1} = -\frac{1}{10} a_2 = -\frac{1}{10}\cdot 0 = 0, \\
a_6 &= a_{4+2} = \frac{2(4-2)(4-4)}{(4+2)(4+1)} a_{4-1} = 0\cdot a_3 = 0, \\
a_7 &\propto a_4 = 0, a_8 \propto a_5 = 0, a_9 \propto a_6 = 0, \ldots
\end{aligned}
\tag{0.4}
$$

Thus the general solution is in fact polynomial: (0.4) implies (0.2) gives

$$
\begin{aligned}
y(x) &= a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 = a_0 + a_1 x + 0\cdot x^2 + a_0 x^3 \\
&= a_0\left(1+x^3\right) + a_1 x.
\end{aligned}
\tag{0.5}
$$

2. According to the relevant theorem, what is the radius of convergence of the series representation of the general solution of (0.1)? Do your results on the previous problem contradict this? Explain.

10 points

<u>Solution</u>

We rewrite (0.1) as

$$
y'' - \frac{6x^2}{2x^3-1} y' + \frac{6x}{2x^3-1} y = 0 \tag{0.6}
$$

which identifies certain functions indicated in the relevant theorem as both having radius of convergence of precisely $\rho = 1/\sqrt[3]{2}$. Thus, guaranteed, the radius of convergence of the series representation of the general solution of (0.1) is at least that big. On the other hand, the actual solution had infinite radius of convergence. But this does not contradict the theorem; the theorem only gives a lower bound on that radius.

3. By use of the definition, compute the Laplace transform $F(s)$ of $f(t) := e^{at}$, where $a = \alpha + i\beta$ is a complex number with real part α and imaginary part β. For which values of s is your formula valid?

10 pts.

Solution:

By definition

$$F(s) := \int_0^{+\infty} e^{-st} f(t)dt = \int_0^{+\infty} e^{-st} e^{at} dt = \int_0^{+\infty} e^{-(s-a)t} dt = \left.\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{+\infty} = 0 - \frac{1}{-(s-a)} \tag{0.7}$$
$$= \frac{1}{s-a},$$

the last steps valid iff $\mathrm{Re}(s-a) > 0 \Leftrightarrow \mathrm{Re}\, s > \mathrm{Re}\, a = \alpha$. If s is considered to only be real (reasonable in this course), this is the restriction $s > \mathrm{Re}\, a = \alpha$.

4. By use of the definition of Laplace transform, show, at least formally, that

$$\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0), \tag{0.8}$$

at least for large enough s (and f of exponential order).

10 points

Solution:

By definition, and using integration by parts (formally), we have

$$\begin{aligned}\mathcal{L}[f'](s) &:= \int_0^{+\infty} e^{-st} f'(t)dt = \int_0^{+\infty} e^{-st} df(t) = e^{-st} f(t)\Big]_0^{+\infty} - \int_0^{+\infty} f(t) de^{-st} \\ &= e^{-\infty} f(\infty) - f(0) - \int_0^{+\infty} f(t)(-s)e^{-st} dt = 0 - f(0) + s\int_0^{+\infty} f(t)e^{-st} dt \\ &= -f(0) + s\mathcal{L}[f](s) = s\mathcal{L}[f](s) - f(0),\end{aligned} \tag{0.9}$$

where we also used f is of exponential order, so that for large enough s

$$\lim_{t\to+\infty} e^{-st} f(t) = 0. \tag{0.10}$$

5. Use (0.8) directly to show that

$$\mathcal{L}[f''](s) = s^2\mathcal{L}[f](s) - sf(0) - f'(0) \tag{0.11}$$

5 points

<u>Solution:</u>

Replacing everywhere in (0.8) f with f' yields

$$\mathcal{L}[f''](s) = \mathcal{L}\left[(f')'\right](s) = s\mathcal{L}[f'](s) - f'(0), \tag{0.12}$$

and using (0.8) in this then gives

$$\begin{aligned}\mathcal{L}[f''](s) &= s\mathcal{L}[f'](s) - f'(0) = s\left(s\mathcal{L}[f](s) - f(0)\right) - f'(0)\\ &= s^2\mathcal{L}[f](s) - sf(0) - f'(0),\end{aligned} \tag{0.13}$$

which gives/is (0.11).

6. ***<u>By use of the Laplace transform</u>***, solve the IVP

$$y' = ay,\ y(0) = y_0 \tag{0.14}$$

10 points

<u>Solution:</u>

Applying the transform to both sides of the ODE, and using (0.8) with the given initial data we get (with linearity)

$$\mathcal{L}[y'](s) = s\mathcal{L}[y](s) - y(0) = s\mathcal{L}[y](s) - y_0 = a\mathcal{L}[y](s) = \mathcal{L}[ay](s), \tag{0.15}$$

for large enough s, i.e.(for those s’s)

$$(s-a)\mathcal{L}[y](s) = y_0 \Leftrightarrow \mathcal{L}[y](s) = \frac{y_0}{s-a} = \mathcal{L}\left[y_0e^{at}\right](s) \Leftrightarrow y(t) = y_0e^{at}. \tag{0.16}$$

(In the last step we used that the Laplace transform is a one-to-one map on the space of continuous functions.)

7. By means of the definition of Laplace transform, at least formally show that for $c > 0$

$$\mathcal{L}\left[f\left(t-c\right)u_c(t)\right](s)=e^{-cs}\mathcal{L}\left[f\left(t\right)\right](s), \tag{0.17}$$

at least for big enough s (when $\mathcal{L}\left[f\left(t\right)\right](s)$ exists for big enough s).

10 points

Solution:

By definition of all the objects involved,

$$\begin{aligned}\mathcal{L}\left[f\left(t-c\right)u_c(t)\right](s)&:=\int_0^{+\infty} f\left(t-c\right)u_c(t)e^{-st}dt=\int_c^{+\infty} f\left(t-c\right)e^{-st}dt\\&=\int_0^{+\infty} f\left(t\right)e^{-s(t+c)}dt=e^{-cs}\int_0^{+\infty} f\left(t\right)e^{-st}dt=:e^{-cs}\mathcal{L}\left[f\left(t\right)\right](s).\end{aligned} \tag{0.18}$$

8. By means of the Laplace transform, solve the I.V.P.

$$y''-2y'+37y=0, \quad y(0)=0, \quad y'(0)=1. \tag{0.19}$$

15 points

Solution:

Taking the Laplace transform of both sides of the ODE in (0.19) and using (0.8) and (0.11) with the data in (0.19) we get

$$\begin{aligned}&s^2\mathcal{L}[y](s)-1-2s\mathcal{L}[y](s)+37\mathcal{L}[y](s)\\&=s^2\mathcal{L}[y](s)-sy(0)-y'(0)-2\left(s\mathcal{L}[y](s)-y(0)\right)+37\mathcal{L}[y](s)=0\\&\Leftrightarrow\\&1=\left(s^2-2s+37\right)\mathcal{L}[y](s)=\left(\left(s-1\right)^2+6^2\right)\mathcal{L}[y](s)\\&\Leftrightarrow\\&\mathcal{L}[y](s)=\frac{1}{\left(s-1\right)^2+6^2}=\frac{1}{6}\frac{6}{\left(s-1\right)^2+6^2}=\mathcal{L}\left[\frac{1}{6}e^t\sin\left(6t\right)\right](s)\\&\Leftrightarrow\\&y\left(t\right)=\frac{1}{6}e^t\sin\left(6t\right).\end{aligned} \tag{0.20}$$

9. By means of the Laplace transform, solve the I.V.P.

$$y'' + y = f(t) := \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & 2\pi \le t. \end{cases}, \quad y(0) = y'(0) = 0. \tag{0.21}$$

Note that after $t = 2\pi$ the undamped system is no longer driven. What is the amplitude of the oscillation after that time?

Feel free to use, if helpful, that

$$\begin{aligned}
\mathcal{L}\left[\frac{1}{2}\sin(t) + \frac{1}{2}t\cos(t)\right](s) &= \frac{1}{2}\left(\mathcal{L}[\sin(t)](s) + \mathcal{L}[t\cos(t)](s)\right) = \frac{1}{2}\left(\frac{1}{s^2+1} + \int_0^\infty e^{-st} t\cos(t)\,dt\right) \\
&= \frac{1}{2}\left(\frac{1}{s^2+1} - \frac{d}{ds}\int_0^\infty e^{-st}\cos(t)\,dt\right) = \frac{1}{2}\left(\frac{1}{s^2+1} - \frac{d}{ds}\mathcal{L}[\cos(t)](s)\right) \\
&= \frac{1}{2}\left(\frac{1}{s^2+1} - \frac{d}{ds}\frac{s}{s^2+1}\right) = \frac{1}{2}\left(\frac{1}{s^2+1} - \frac{\left(s^2+1\right) - s\cdot 2s}{\left(s^2+1\right)^2}\right) \\
&= \frac{1}{2}\frac{2s^2}{\left(s^2+1\right)^2} = \frac{s^2}{\left(s^2+1\right)^2}, \\
\mathcal{L}\left[\frac{1}{2}\sin(t) - \frac{1}{2}t\cos(t)\right](s) &= \frac{1}{2}\left(\mathcal{L}[\sin(t)](s) - \mathcal{L}[t\cos(t)](s)\right) = \frac{1}{2}\left(\frac{1}{s^2+1} - \int_0^\infty e^{-st} t\cos(t)\,dt\right) \\
&= \frac{1}{2}\left(\frac{1}{s^2+1} + \frac{d}{ds}\int_0^\infty e^{-st}\cos(t)\,dt\right) = \frac{1}{2}\left(\frac{1}{s^2+1} + \frac{d}{ds}\mathcal{L}[\cos(t)](s)\right) \\
&= \frac{1}{2}\left(\frac{1}{s^2+1} + \frac{d}{ds}\frac{s}{s^2+1}\right) = \frac{1}{2}\left(\frac{1}{s^2+1} + \frac{\left(s^2+1\right) - s\cdot 2s}{\left(s^2+1\right)^2}\right) \\
&= \frac{1}{2}\frac{2\left(s^2+1\right) - 2s^2}{\left(s^2+1\right)^2} = \frac{1}{\left(s^2+1\right)^2}.
\end{aligned} \tag{0.22}$$

15 points

<u>Solution:</u>

For positive t, rewrite the r.h.s. of the equation as

$$
\begin{aligned}
f(t) &= \sin t\left(1-u_{2\pi}(t)\right) = \sin t - \sin t u_{2\pi}(t) = \sin t - \sin\left(t-2\pi\right)u_{2\pi}(t) \\
&= \mathcal{L}^{-1}\left[\frac{1}{s^2+1} - e^{-2\pi s}\frac{1}{s^2+1}\right](t) = \mathcal{L}^{-1}\left[\frac{1-e^{-2\pi s}}{s^2+1}\right](t).
\end{aligned} \tag{0.23}
$$

So taking the Laplace transform of both sides of the ODE with (trivial) data we get (by shift theorems) that

$$
\left(s^2+1\right)\mathcal{L}\left[y\right](s) = \frac{1-e^{-2\pi s}}{s^2+1} \Leftrightarrow \mathcal{L}\left[y\right](s) = \frac{1-e^{-2\pi s}}{\left(s^2+1\right)^2} = \mathcal{L}\left[h(t)-h(t-2\pi)u_{2\pi}(t)\right](s)
$$

$$
\Leftrightarrow
$$

$$
y(t) = h(t) - h(t-2\pi)u_{2\pi}(t),
$$

where, by hint (0.22),

$$
h(t) = \frac{1}{2}\sin(t) - \frac{1}{2}t\cos(t). \tag{0.24}
$$

Note that for $t > 2\pi$ we have

$$
\begin{aligned}
y(t) &= h(t) - h(t-2\pi)u_{2\pi}(t) = h(t) - h(t-2\pi) = \frac{1}{2}\sin(t) - \frac{1}{2}t\cos(t) - \left(\frac{1}{2}\sin(t-2\pi) - \frac{1}{2}\left(t-2\pi\right)\cos(t-2\pi)\right) \\
&= \frac{1}{2}\sin(t) - \frac{1}{2}t\cos(t) - \left(\frac{1}{2}\sin(t) - \frac{1}{2}\left(t-2\pi\right)\cos(t)\right) = -\pi\cos(t).
\end{aligned} \tag{0.25}
$$

Thus the amplitude of the oscillation after $t = 2\pi$ is clearly π.

10. By means of the Laplace transform, solve the I.V.P.

$$
y'' + y = \delta(t-1), \quad y(0) = y'(0) = 0.
$$

10 Points

<u>Solution</u>

Taking the Laplace transform of both sides of the ODE with (trivial) data we get (by shift and other theorems) that

$$\left(s^2+1\right)\mathcal{L}\left[y\right](s)=e^{-s}\Leftrightarrow\mathcal{L}\left[y\right](s)=\frac{e^{-s}}{s^2+1}=\mathcal{L}\left[\sin(t-1)u_1(t)\right](s)$$
$$\Leftrightarrow \tag{0.26}$$
$$y(t)=\sin(t-1)u_1(t).$$

11. By means of the Laplace transform, express the solution of the following IVP in terms of a convolution integral:

$$y''+y=g(t),\ \ y(0)=y'(0)=0. \tag{0.27}$$

10 Points

Solution

Taking the Laplace transform of both sides of the ODE with (trivial) data we get

$$\left(s^2+1\right)\mathcal{L}\left[y\right](s)=\mathcal{L}\left[g\right](s)\Leftrightarrow\mathcal{L}\left[y\right](s)=\frac{1}{s^2+1}\cdot\mathcal{L}\left[g\right](s)=\mathcal{L}\left[\left(\sin * g\right)(t)\right](s)$$
$$\Leftrightarrow \tag{0.28}$$
$$y(t)=\left(\sin * g\right)(t)=\int_0^t\sin\left(t-\tau\right)g(\tau)d\tau.$$

12. A system of 2, first order, linear, homogeneous, ordinary differential equations appears as

$$\mathbf{x}'=P(t)\mathbf{x}. \tag{0.29}$$

Suppose, over the relevant interval(s), we have the following fundamental set of solutions of (0.29):

$$\left\{\mathbf{x}^{(1)}(t),\mathbf{x}^{(2)}(t)\right\}=\left\{\begin{bmatrix}2t\\1\end{bmatrix},\begin{bmatrix}t+1\\1\end{bmatrix}\right\}. \tag{0.30}$$

What is the Wronskian of this fundamental set? Where is $P(t)$ continuous? Explain.

10 points

Solution

We have

$$W\left[\mathbf{x}^{(1)},\mathbf{x}^{(2)}\right](t)=\det\begin{bmatrix}2t & t+1\\ 1 & 1\end{bmatrix}=2t-(t+1)=t-1, \tag{0.31}$$

which vanishes at, and only at, $t=1$. Thus, according to the relevant theorem, $P(t)$ may be continuous everywhere but $t=1$, where it must in fact be singular.

13. The general solution of a system of 2, first order, linear, homogeneous, constant coefficient ordinary differential equations appears as

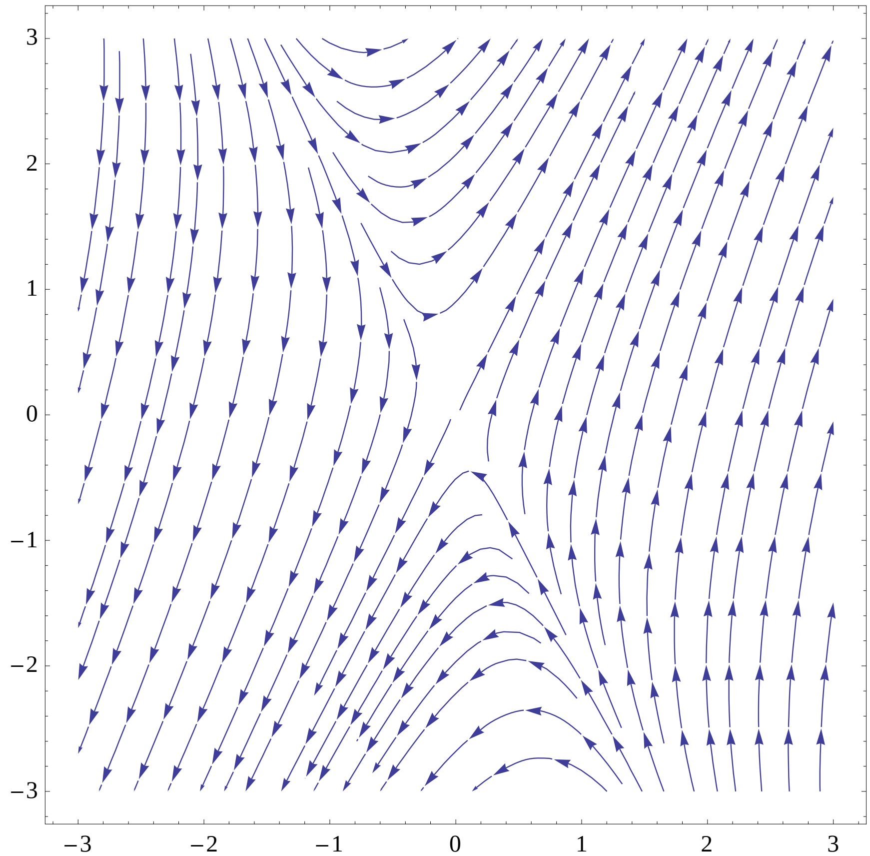
$$\mathbf{x}(t)=c_1\begin{bmatrix}1\\ 2\end{bmatrix}e^{3t}+c_2\begin{bmatrix}1\\ -2\end{bmatrix}e^{-t}. \tag{0.32}$$

Sketch a phase portrait of these dynamics.

5 points

Solution

Mathematica gives the following output with input $\mathrm{StreamPlot}[\{x+y,4x+y\},\{x,-3,3\},\{y,-3,3\}]$ (which input comes from thinking about what the matrix A must be in writing the system as $\mathbf{x}'=A\mathbf{x}$):



The picture can be obtained "from scratch" by realizing that (0.32) indicates that

$$\mathbf{x}^{(1)}(t) := \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}, \quad \text{and} \quad \mathbf{x}^{(2)}(t) := \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} \tag{0.33}$$

are special solutions of the indicated equation, special in that trajectories given by them are (separately) straight lines through the origin, i.e. subspaces of $\mathbb{R}^2$, the first of which having solutions emanating (as in the above plot) away from the origin (towards infinity), the second of which having solutions tending to the origin (from infinity), and that other trajectories/integral curves arise from extrapolating in the indicated way between these special solutions.